IISER PUNE

SPRING 2015 MATHEMATICS COMPREHENSIVE EXAM

ALGEBRA

Duration: 3 hours **Maximum marks: 120**

Q.1 Let p be a prime and $\mathbb{F}_q$ be the finite field of size $q = p^m$ for some integer $m > 0$.

(a) Suppose χ is the set of all pairs (L, P) where L is a linear subspace of ${\mathbb{F}_q}^3$ of dimension 1 and P is a linear subspace of dimension 2, satisfying $L \oplus P \cong {\mathbb{F}_q}^3$. Define an action of $G = \mathbb{GL}(3, \mathbb{F}_q)$ on χ by $T \cdot (L, P) = (T(L), T(P))$. Prove that,

(i) The action of G on χ is transitive.

(ii) Find the stabilizers of elements of χ.

[**10 marks**]

(b) Show that p divides $|G|$. Describe all p-Sylow subgroups of G. [**10 marks**]

Q.2 Let F be an algebraically closed field. Let V be a finite dimensional vector space over F. Let $T : V \to V$ be a linear transformation. Consider the $F[t]$-module structure on V given by T. Prove that if T has distinct eigenvalues, then V is a cyclic $F[t]$-module. Taking $V = F^2$, construct a counterexample to the converse. [**20 marks**]

Hint : You may use the following fact.

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Q.3 Suppose R is a UFD. If $(i, j) = 1$ for two positive natural numbers i and j, prove that $X^i - Y^j$ is irreducible in the polynomial ring $R[X, Y]$. **[20 marks]**

Hint : Show that $R[X, Y]/(X^i - Y^j) \cong R[T^j, T^i]$

Q.4 Let p be a prime and $\mathbb{F}_q$ be the finite field of size $q = p^m$ for some integer $m > 0$. Let τ be the map defined by $\tau(x) = x^p \quad \forall\, x \in \mathbb{F}_q$.

(a) Show that τ is an automorphism of $\mathbb{F}_q$ which is $\mathbb{F}_p$-linear. **[5 marks]**

(b) Show that $\mathbb{F}_q$ is a Galois extension over $\mathbb{F}_p$ and its Galois group $G = \mathrm{Gal}(\mathbb{F}_q : \mathbb{F}_p)$ is the cyclic group generated by τ. **[10 marks]**

(c) Find the characteristic polynomial of τ (as a $\mathbb{F}_p$-linear map on $\mathbb{F}_q$). **[5 marks]**

Q.5 Suppose m and n are coprime integers and let p be any other integer. Consider the short exact sequence of $\mathbb{Z}$ modules

$$0 \longrightarrow \frac{\mathbb{Z}}{p\mathbb{Z}} \xrightarrow{\alpha} \frac{\mathbb{Z}}{(mp)\mathbb{Z}} \xrightarrow{\beta} \frac{\mathbb{Z}}{m\mathbb{Z}} \longrightarrow 0$$

where α is such that $\alpha(1) = m$ and β is the usual quotient by the ideal (p). Will this sequence remain exact after applying the functor $\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z}/n\mathbb{Z})$? **[20 marks]**

Q.6 Let A be a commutative local ring and $\{M_i\}_{i=0}^{n}$ be finitely generated free A-modules. For each $0 \leq j \leq n$, let b_j denote the free rank of M_j as A-module. Suppose that there is an exact sequence as follows:

$$0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to 0$$

Show that

$$b_0 + b_2 + \cdots \quad = \quad b_1 + b_3 + \cdots$$

[20 marks]